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Discriminated dimensional analysis of the energy equation: Application to laminar forced convection along a flat plate

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Abstract

In contrast to "classical" dimensional analysis, whose application is widely described in heat transfer textbooks despite its poor results, the less well-known discriminated dimensional analysis approach provides a deeper insight into the physical problems involved and much better results in all cases where it is applied. The basis of this technique is firstly used to test the dimensional homogeneity of the energy equation for incompressible fluids. It is then applied to the laminar forced convection on flat plates to determine the characteristic lengths of the problem, drag forces and heat transfer coefficient. Neither the classical Reynolds and Nusselt numbers nor the drag coefficient are relevant dimensionless parameters for the discriminated dimensional analysis and they do not play a separate (independent) role in the solution of this kind of problem. Furthermore, the dimensionless groups that really play a separate role are obtained with this technique. The apparent equivalence between dimensional analysis and scale analysis is discussed.

Keywords: Discrimination; Dimensional analysis; Energy equation; Laminar forced convection

1. Introduction

Most text-books on conduction and convection heat transfer, both classical [1,2] and modern [3–10], make use of the results obtained by dimensional analysis, which we name "classical dimensional analysis" (CDA, hereinafter), to deduce, from the large number of variables that are generally involved in this kind of problem, the dimensionless groups of variables as a function of which the solutions may be expressed. This is, undoubtedly, one of the main advantages of CDA, since the number of dimensionless groups which fully describe the problem is much smaller than the number of non-dimensionless physical quantities that take part in it.

However, it is important to mention that some prestigious text-books used in teaching and investigation [11,12] scarcely make reference to dimensional analysis. Bejan [12], for example, extensively uses scale analysis (SA, hereinafter) to produce order-of-magnitude estimates of the quantities of interest, a goal beyond the scope of CDA. According to Bejan, "scale analysis is a technique often confused either with the CDA or with the nondimensionalization process of the governing equation, so extended at the research literature". As regards the differences and similarities between discriminated dimensional analysis (DDA, hereinafter) and scale analysis, these are discoursed in a paragraph at the end of this paper.

Nevertheless, in spite of the advanced level of the above mentioned text-books, the techniques of dimensional analysis are simply applied using the typical "non-discriminated" dimensional bases. Researchers into convection heat transfer use a dimensional base made up of four or five basic quantities (length, mass, time and temperature) usually expressed by the symbols (L, M, T, θ) or (L, M, T, Q, θ), including (or not) the heat quantity, without formally justifying such an inclusion from the point of view of dimensional analysis theory. For example, McAdams [1] uses the base (L, M, T, θ), whereas Chapman [2] and Gröber [13] use the base (L, M, T, Q, θ). This controversy also exists among

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Nomenclature

CDA	classical dimensional analysis	
c_p	specific heat	$\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$
DDA	discriminated dimensional analysis	
Fo	Fourier number, $= \alpha t l_0^{-2}$	
h	specific enthalpy	$\mathrm{J\cdot kg^{-1}}$
	heat transfer coefficient	$\mathrm{W\cdot m^{-2}\cdot K^{-1}}$
k	thermal conductivity	$\mathrm{W\cdot m^{-1}\cdot K^{-1}}$
Nu	Nusselt number, $= hl/k$	
p	pressure	$\mathrm{N\cdot m^{-2}}$
Pr	Prandtl number, $= \nu/\alpha$	
SA	scale analysis	
s	specific entropy	$\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$
t	time	s
u	specific internal energy	$\mathrm{J\cdot kg^{-1}}$
v	velocity	$\mathrm{m\cdot s^{-1}}$
W	mechanical energy	J
x, y, z	spatial coordinates	m

Greek symbols

α	thermal diffusivity	$\mathrm{m^2\cdot s^{-1}}$
δ	boundary layer thickness	m
δ_{ik}	Kronecker delta	
μ	dynamic viscosity	$\mathrm{kg\cdot m^{-1}\cdot s^{-1}}$
ν	cinematic viscosity	$\mathrm{m^2\cdot s^{-1}}$
θ	temperature	K
ρ	fluid density	$\mathrm{kg\cdot m^{-3}}$
ρc_p	volumetric heat capacity	$\mathrm{J\cdot m^{-3}\cdot K^{-1}}$
$\boldsymbol{\sigma}'$	component of the shear stress tensor associated to the pressure forces	$\mathrm{N\cdot m^{-2}}$
$\boldsymbol{\sigma}$	shear stress tensor	$\mathrm{N\cdot m^{-2}}$
ξ	viscosity coefficient	$\mathrm{kg\cdot m^{-1}\cdot s^{-1}}$
$\propto$	proportional	
$\sim$	order of magnitude	
[ ]	denote the dimensional equation of the enclosed quantity	

Italic letters (used to designate the quantities belonging to the dimensional base)

L	length
M	mass
T	time
Q	heat
θ	temperature

Subscripts

x, y, z	spatial directions

classical authors specialized in dimensional analysis, such as Bridgman [14], who uses the base of McAdams and Huntley [15] who, in turn, use that of Chapman and Gröber.

In an attempt to increase the number of dimensions of the base and, as a consequence, reduce the number of dimensionless groups in the problem under study, Mills [8] established a distinction between what he calls "*simple dimensional analysis*" and "*vectorial dimensional analysis*", mentioning that with the last term the lengths measured in different directions (coordinates) in space can be adopted as independent dimensions. Historically, the idea of considering spatial dimensions as "different" dimensions was first proposed by Williams [16] in 1892; later, in the second half of XX century, Huntley [15] used the idea developed by Williams, introducing the concept of "the method of the components of fundamental dimensions", and Runge [17] called it "vectorial dimensions". Palacios [18] deals rigorously with this subject and sets the basis for applying DDA, including a large number of examples in his book. He says "DDA augments the number of equations in the problem but it brings about a diminution in the number of dimensionless groups and the solution becomes more precise".

The consequence of using dimensional analysis in the classical sense (no spatial discrimination) of Bridgman [14] and Langhaar [19] poses two problems: (i) it may introduce into the solution certain non-dimensionless groups, which are the ratio between purely geometric quantities, such as the ratios between characteristic lengths (the so-called "form factors"), and, furthermore, excessively increase the number of non-dimensionless groups, which renders application of this technique useless, since many of these groups do not really take part as independent groups in the solution (as occurs in fin-wall assembly problems, where many geometrical variables exist); (ii) it provides dimensionless groups that do not play an independent role in the problem. In general these disadvantages are sidestepped by experienced researchers without explanation (from the CDA point of view). As we see below, the use of DDA removes these groups immediately so that, for example, the form factors are not dimensionless groups using the discrimination. In spite of all this, some books, both of general [18] and specific [20] interest, and other scientific works [21] devote a great deal of effort to the formal application of DDA to engineering problems, particularly in heat transfer.

In this work, the dimensional homogeneity of the energy equation is studied first from a DDA perspective. Next, as an application, the problem of laminar forced convection along a flat plate with negligible friction dissipation is studied, providing the characteristic lengths and the drag and heat transfer coefficients. It is shown that, from the DDA perspective, the drag coefficient and the well-known Reynolds and Nusselt numbers are not relevant dimensionless parameters and do not play an independent role in the solutions. DDA directly provides the truly dimensionless groups that take part in the solution. Finally, similarities between scale analysis and DDA are discussed.

2. Discriminated dimensional analysis versus classical dimensional analysis

The variables that take part in a problem should form a "physical relationship" that is independent of the system of units chosen to measure them, a condition that is termed "dimensional homogeneity". The consequence of this condition is that the addends of each of the partial differential equations that define the mathematical model of the problem must be dimensionally homogeneous. Now, the dimensionless groups relevant to the problem may be deduced either directly, by combining the variables adequately, using the typical techniques of CDA (this procedure is generally used when the equations of the problem are unknown), or by the nondimensionalization of such equations by defining the appropriate dimensionless quantities. In the first case, the dimensional equations of all the variables (velocity, location, temperature, pressure, ...) given by their definition, and those of all the parameters (viscosity, conductivity, specific heat, ...) deduced from the equations or laws by which they are defined, must be known. So far, the above paragraph applies both to CDA and DDA.

The essential difference between CDA and DDA [18] is that in the latter, the spatial coordinates (vectors) and all the vectorial quantities have dimensional equations in which these spatial coordinates appear explicitly. That is, in DDA there exists the possibility of using different units of measurement for the three coordinates or for the three vectorial components of any quantity of the problem, making these quantities dimensionally independent [18, p. 72]. Of course, other geometries (cylindrical, spherical or intrinsic coordinates) may be used, depending on the problem under study. Also, new quantities (such as angles, surfaces, etc.), whose inclusion in the base was not considered before, can now be included [20, vol. 4]. For example, the angle in the base provides a new dimensional equation for the angular velocity, ω, different from the classical $[\omega] = T^{-1}$, and the surface in the base may be adequate when all the directions contained in it are physically equivalent (degenerated) for the particular problem. Moreover, the mass may also be discriminated to distinguish the inertial effect from the amount of matter [15], and two time scales can be included in the dimensional bases if two simultaneous phenomena of different duration take place in the same problem, some quantities or parameters being associated to one duration and some quantities to the other.

The main contribution of DDA is that it reduces the number of dimensionless parameters (for a given number of variables in the relevant list) that play an independent role in the problem. The discrimination increases the number of equations in the problem of dimensional analysis and decreases the number of independent dimensionless parameters, which makes the solution more precise. However, that is not the only advantage. The assumption of only one length in CDA prevents distinction, for example, between the quantity of shear stress (or inertia forces) associated to different spatial directions. The degeneration of length associated to spatial directions, which is inherent in classical dimensional analysis, permits us to form numbers such as *Re* or *Nu*, or coefficients such as the drag-friction. These numbers or coefficients are really non-dimensionless parameters whose dimensional equations and real meaning (far from that generally assumed in many text books and handbooks) are provided by DDA.

With regard to nondimensionalization of the differential equations, which is carried out whenever possible to determine universal solutions, discrimination of the spatial directions prevents the degeneration alluded to above. Dimensionless coordinates are formed by relating spatial coordinates with characteristic lengths of the problem in the same direction (if they exist). Again, the discriminated dimensionless groups formed (following this procedure) are well defined and have the physical meaning associated to the rate or balance between the corresponding addends of the differential equation. Obviously, the number of relevant dimensionless groups obtained from the nondimensionalizing the equations and from the application of the techniques proper to DDA is the same.

3. Dimensional homogeneity of the energy equation for incompressible fluids using DDA

The energy equation in fluids results from the balance between: (i) time changes in the internal and mechanical energy contained in the mass unit, $\partial(\rho \mathbf{v}^2/2 + \rho u)/\partial t$, (ii) energy flux density associated to the mass transfer, $\mathbf{div}[(\rho \mathbf{v}^2/2 + h)]$, (iii) energy flux density associated to the internal friction processes due to viscosity, which take place either in the whole fluid, $\mathbf{grad}(\theta) = 0$, or with a non-uniform spatial distribution, $\mathbf{div}[-\mathbf{v}\boldsymbol{\sigma}']$, and (iv) heat flux density derived from conduction processes, dependent on $\mathbf{grad}(\theta)$, derived from the existence of either dissipation processes or heat sources within the fluid; this flux, developed in $\mathbf{grad}(\theta)$ series, may be approximated by $\mathbf{div}[-k\,\mathbf{grad}(\theta)]$. So the energy balance may be written as

$$\partial\big(\rho \mathbf{v}^2/2 + \rho u\big)/\partial t = -\,\mathbf{div}\big[\big(\rho \mathbf{v}^2/2 + h\big) - \mathbf{v}\boldsymbol{\sigma}' - k\,\mathbf{grad}(\theta)\big] \qquad (1)$$

Taking into account certain thermodynamic relations, conservation laws and the Navier–Stokes equations, Eq. (1) takes the form:

$$\rho\theta\big[\partial s/\partial t + \mathbf{v}\,\mathbf{grad}(s)\big] = \sigma'_{ik}(\partial v_i/\partial x_k) + \mathbf{grad}\big[k\,\mathbf{grad}(\theta)\big] \qquad (2)$$

In this equation, $[\partial s/\partial t + \mathbf{v}\,\mathbf{grad}(s)]$ is the total change (local and convective) of the specific entropy of the fluid and $\rho\theta[\partial s/\partial t + \mathbf{v}\,\mathbf{grad}(s)]$ is the specific storage heat increased by time unit. The latter is due to the energy dissipated by the viscous processes, $\sigma'_{ik}\,(\partial v_i/\partial x_k)$, plus the energy introduced in the fluid mass by thermal conduction, $\mathbf{grad}[k\,\mathbf{grad}(\theta)]$.

The shear tensor of the fluid, $\boldsymbol{\sigma}$, is composed of two addends, one associated to the friction forces and one associated to the pressure forces, $\sigma_{ik} = \sigma'_{ik} - p\delta_{ik}$. Using the more general form for σ'_{ik}

$$\sigma'_{ik} = \mu\big[(\partial v_i/\partial x_k) + (\partial v_k/\partial x_i) - (2/3)\delta_{ik}(\partial v_l/\partial x_l)\big] + \xi\delta_{ik}(\partial v_l/\partial\partial x_l) \tag{3}$$

which takes advantage of the fact that the expression between square brackets reduces to zero by contraction ($i = k$). Eq. (2) for incompressible fluids reduces to

$$\partial\theta/\partial t + \mathbf{v}\,\mathbf{grad}(\theta) = \alpha\Delta(\theta) + (\nu/2c_p)\big[(\partial v_i/\partial x_k) + (\partial v_k/\partial x_i)\big]^2 \tag{4}$$

with $\alpha = k/(\rho c_p)$ and $\nu = \mu/\rho$. For an isotropic and homogeneous medium at rest, whose properties are temperature independent, the above equation takes the form of the Fourier conduction equation $\partial\theta/\partial t = \alpha\Delta(\theta)$.

Two cases may be distinguished in the study of heat transfer processes between solid surfaces and fluids in motion: (i) heat transfer with energy conversion from mechanical to thermal, in which the last term of Eq. (4) is of the same order of magnitude as the other terms, and (ii) heat transfer without energy conversion, in which the last term is negligible. From the point of view of dimensional analysis, Madrid [20,21] formally justifies that, in the first case, the number of dimensions in the dimensional base is four, the most extended base being (L, Q, T, θ), whereas in the second case, the number of dimensions is five, the base increasing by inclusion of the mass quantity, (L, M, Q, T, θ). In the last case, the dimensional equations of each of the terms of Eq. (4) are

- $[\partial\theta/\partial t] = \theta T^{-1}$;
- $[\mathbf{v}\,\mathbf{grad}(\theta)] = [\sum_{i,j,k}\{v_i(\partial/\partial x_i)\}\theta] = \theta T^{-1}$, since $[v_i(\partial/\partial x_i] = T^{-1}$;
- The dimensional equation of each of the addends of $\Delta(\theta)$ is not the same. Nevertheless, the addends of $\alpha\Delta(\theta)$ have the same dimensional equation since α is a tensorial quantity and the components of the tensor have different dimensional equations in DDA.

Indeed, if discrimination is applied to the length associated to the direction of the thermal gradient, $L_{\mathrm{grad}(\theta)}$, and to the surface normal to that direction, namely $S_{\mathrm{grad}(\theta)}$, the dimensional equations of k and ρc_p are:

$$[k] = [Q]\big[t^{-1}\big]/\big([S][\theta/n]\big) = QT^{-1}S^{-1}_{\mathrm{grad}(\theta)}\theta L_{\mathrm{grad}(\theta)}$$

$$[\rho c_p] = \big[mV^{-1}c_p\big] = \big[V^{-1}\big][Q]/[\theta] = S^{-1}_{\mathrm{grad}(\theta)}L^{-1}_{\mathrm{grad}(\theta)}Q\theta^{-1}$$

where V denotes the volume of the fluid, $[V] = S_{\mathrm{grad}(\theta)} \times L_{\mathrm{grad}(\theta)}$. In this way,

$$[\alpha] = [k]/[\rho c_p] = L^2_{\mathrm{grad}(\theta)}T^{-1}$$

the typical dimensions of a diffusion coefficient associated to the heat diffusion in the direction of $L_{\mathrm{grad}(\theta)}$.

Applying the previous discussion to the discriminated base $(L_x, L_y, L_z, Q, T, \theta, M)$, we can write:

$$[\alpha_x] = (L_{\mathrm{grad}(\theta),x})^2T^{-1}$$

$$[\alpha_y] = (L_{\mathrm{grad}(\theta),y})^2T^{-1}$$

$$[\alpha_z] = (L_{\mathrm{grad}(\theta),z})^2T^{-1}$$

and "$\alpha\Delta(\theta)$" may be expressed in matrix form

$$(\alpha_x \quad \alpha_y \quad \alpha_z)^{\mathrm{T}}\big(\partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad \partial^2/\partial z^2\big)\theta$$

whose addends are dimensionally homogeneous and have the dimensional equation θT^{-1}, which is the same as the dimension equation for "$\partial\theta/\partial t$" and "$\mathbf{v}\,\mathbf{grad}(\theta)$".

For the heat transfer involving energy conversion (friction heat dissipation) the correct dimensional base is (L, Q, T, θ) or its corresponding discriminated bases [21]. It is easy to demonstrate that the first three terms of Eq. (4) are homogeneous, their dimensional equation being θT^{-1}. However, deduction of the dimensional equation of the last term presents greater difficulty. The kinematic viscosity (which characterizes the momentum diffusion) has the typical dimension of a diffusion coefficient, $L^2_\perp/T$, with $L_\perp$ the direction in which the momentum is diffused (normal to the velocity vector $\mathbf{v}$).

Indeed, we will make the discrimination choosing two directions, one parallel to the fluid velocity (which coincides with the direction of the friction force, $L_\parallel$) and the other normal to the friction surface, $L_\perp$. To complete the base, the friction surface, $S_\perp$, is added. In this base $(L_\parallel, L_\perp, S_\perp, Q, T, \theta)$ the dimensional equations of the force, mass and density [19] are,

$$[W] = [Q] = Q, \qquad [F] = [W]/[r] = L_\parallel^{-1}Q$$

$$[m] = [F/a] = L_\parallel^{-1}Q/L_\parallel T^{-2} = L_\parallel^{-2}QT^{-2}$$

$$[\rho] = [m/V] = L_\parallel^{-2}QT^{-2}/L_\perp S_\perp = L_\parallel^{-2}L_\perp^{-1}S_\perp^{-1}QT^2$$

so that the dimensional equation of the dynamic viscosity μ, from the Newton law, $\mathbf{F} = \mu S(\partial v/\partial n)$, is:

$$[\mu] = [F]/[S][v/n] = L_\parallel^{-1}Q(S_\perp)^{-1}\big(L_\parallel T^{-1}/L_\perp\big)^{-1} = L_\parallel^{-2}L_\perp S_\perp^{-1}QT$$

and that of kinematic viscosity, ν,

$$[\nu] = [\mu]/[\rho] = L^2_\perp T^{-1}$$

which has the same tensorial character as that of the thermal diffusivity. In rectangular coordinates,

$$[\nu_i] = L_i^2T^{-1}, \quad i = x, y, z$$

and, using the above base $(L_\parallel, L_\perp, S_\perp, Q, T, \theta)$

$$\big[(\partial v_i/\partial x_k)^2\big] = L^2_\parallel L^{-2}_\perp T^{-2}$$

$$[c_p] = [Q/m\theta] = L^2_\parallel T^{-2}\theta^{-1}$$

$$\big[(\nu_i/2c_p)\big] = L_\parallel^{-2}L^2_\perp T\theta$$

giving

$$\big[(\nu_i/2c_p)(\partial v_i/\partial x_k)^2\big] = \theta T^{-1}$$

In conclusion, Eq. (4) is also homogeneous in the discriminated base.

4. Application. The problem of laminar forced convection along an isothermal flat plate with negligible dissipation

4.1. Determination of characteristic lengths and their meanings

The relevant list for the mechanical (fluid) problem is formed by the quantities:

- typical steady velocity far from the boundary layer, v_0,
- dynamic viscosity of the fluid, μ,
- density of the fluid, ρ,
- length of the flat plate, l_0.

In addition, the local positions (independent spatial coordinates) should be included in the relevant list if we wish to investigate local unknown dimensionless groups, an aspect that is not considered here.

Firstly we will investigate what CDA and DDA can tell as about the above list of quantities. The use of CDA, which assumes the base (L, T, M) provides only one dimensionless group, namely $\pi_1 = \rho v_0 l_0 \mu^{-1} = v_0 l_0 \nu^{-1} = Re_{l_0}$, the known Reynolds number, which, strictly according to analysis dimensional theory, will participate in the solution of the problem. As regards DDA, the dimensional equations of the quantities for the base $(L_\parallel, L_\perp, S_\perp, T, M)$, have the exponents shown in the columns of Table 1 (exponents of μ are derived from the Newton law of viscosity). Although a base with three discriminated lengths is applicable, we adopt a complete base that includes the sliding viscous surfaces because this variable is more closely related with the physical problem.

As is well known, the number of the independent dimensionless groups, i, that may be derived from the variables of the relevant list, n, is $i = n - H$, where H is the rank of the matrix of the dimensional exponents [18]. From Table 1, $n = 4$, $H = 4$ and $i = 0$. In consequence, DDA does not permit Re_{l_0} $(= \rho v_0 l_0/\mu)$ to play an independent role in the problem. That is, from the point of view of DDA, Re_{l_0} is a non-dimensionless number, whose dimensional equation is $[Re_{l_0}] = L_\parallel^2 L_\perp^{-2}$. The same result would appear if l_0 were normal to the sliding surfaces (as occurs in laminar duct flow in tubes, where l_0 is the radius or the diameter of the tube); in this case, the only change in Table 1 would be the exponent of the dimensional equation of l_0, which would become $[l_0] = L_\perp$, resulting in a different dimensional equation for Re_L, $[Re_{l_0}] = L_\parallel L_\perp^{-1}$.

Table 1
Fluid mechanical quantities and their dimensional exponents

	Relevant list			
	v_0	l_0	ρ	μ
$L_\parallel$	1	1		
$L_\perp$			−1	1
$S_\perp$			−1	−1
T	−1			−1
M			1	1

We have now verified that the classical Reynolds number, Re_{l_0} does not play an independent role when using DDA. Now, from the relevant list of Table 1, it is a straightforward task to obtain a characteristic length l^*, of dimension $[l^*] = L_\perp$, that is $l^* \propto (\mu l_0/\rho v_0)^{1/2}$. In this case, l^* is what we call a "hidden" quantity since it is not explicitly included in the relevant list because it does not belong to the list of known variables. Indeed, introducing the quantity l^* into Table 1, $[l^*] = L_\perp$, the rank of the new matrix is $H = 4$ and $i = n - H = 1$. If ε is the exponent of each variable in the dimensionless group, the solution of the equations

$$\varepsilon_{v_0} + \varepsilon_{l_0} = 0$$
$$-\varepsilon_\rho + \varepsilon_\mu + \varepsilon_{l^*} = 0$$
$$-\varepsilon_\rho - \varepsilon_\mu = 0$$
$$-\varepsilon_{v_0} - \varepsilon_\mu = 0$$
$$\varepsilon_\rho + \varepsilon_\mu = 0$$

provides only one dimensionless group $\pi_d = \rho v_0 l^{*2}/(\mu l_0)$ or $\pi_d = Re_{l_0}(l^*/l_0)^2$ in terms of the classical Re_{l_0} number [18].

But, what is the physical meaning of l^*? Of course l^* is linked to the variables that define the problem and which are linked to the physical process and to the hypothesis assumed. In short, the meaning of l^* must be derived from the dimensionless group associated to it, and this group (and all the dimensionless groups that may be deduced from the relevant list) is a balance of certain quantities (mass, forces or energy) of the problem. For the quantities of Table 1, $\pi_d = \rho v_0 l^{*2}/(\mu l_0)$ is the balance between inertial and viscous forces, so that l^* is the limit (in the $L_\perp$ direction) of the region in which this balance takes place, that is, the thickness of the boundary layer. $l^* = \delta_v$ according to the nomenclature extensively used in textbooks. Only within this region may the above balance be defined since the fluid outside this region is not subjected to inertial or viscous forces.

The meaning of $Re_{l_0} = \rho v l_0/\mu$ in the laminar forced convection along flat plates is clear, $Re_{l_0}^{1/2} \propto l_0/l^*$, the ratio between the plate length and the boundary layer thickness. Bejan [12] assigns the same meaning to $Re_{l_0}^{1/2}$ basing his arguments on scale analysis arguments. For other types of flow (internal tube duct flow, transverse flow, etc.), the dimensionless group that contains the hidden quantity, l^*, has the same meaning of balance between inertial and viscous forces, but neither l^* nor Re_{l_0} (with l_0 the diameter of the tube) has the meaning given in the flat plate problem.

4.2. Determination of the force exerted by the fluid on the plate (drag force)

This quantity predicts the total drag exerted by the stream on the plate. This force translates into the pressure drop per unit of plate length and hence into the pumping power necessary to keep the fluid stream flowing. The relevant list increases by the quantity of the shear stress (skin friction) experienced by the plate, τ, to which in accordance with its definition, $\tau = \mu(\partial v/\partial n)$, DDA gives the dimensional equation $[\tau] = ML_\parallel T^{-2} S_\perp^{-1}$. Table 2 shows the dimensional exponents of the new relevant list. From this and applying the procedure mentioned in 4.1, the only dimensionless group that may be formed is $\pi_d = \tau^2 l_0/(\rho v_0^3 \mu)$, from which we deduce $\tau \propto (\rho v_0^3 \mu/l_0)^{1/2}$. Most text books and handbooks define the skin friction coefficient as $C_f = \tau/(\rho v_0^2/2)$; substituting C_f in the above dimensionless group, the known proportion between the classical Reynolds number and the skin friction coefficient appears, $C_f = Re_{l_0}^{-1/2}$. Again, despite the fact that C_f is a dimensionless number in CDA and that some books [6] give it the meaning of "ratio of surface shear stress to free stream kinetic energy", from the point of view of DDA, C_f is neither a relevant dimensionless parameter nor can its meaning be the balance between (any) quantities associated to the same region of the fluid. The dimension of C_f is $[C_f] = L_\perp L_\parallel^{-1}$, which is the ratio between the boundary layer thickness and the plate length. Otherwise, an easy inference leads to the expression of the ratio of surface shear stress to free stream kinetic energy, that is $C_f l_0/l^*$ or $C_f Re_{l_0}^{1/2}$ (a dimensionless group from the perspective of DDA).

4.3. Determination of the heat transfer coefficient

The heat transfer coefficient is a measure of the resistance to the heat transfer from the fluid stream to the plate or vice versa. The relevant list of quantities and their dimensional exponents, when friction dissipation is negligible, is shown in Table 3.

The general case ($0 < Pr < \infty$). From Table 3 and again applying the procedure mentioned in 4.1, the dimensionless groups that may be formed are

$$\pi_{d,1} = hk^{-1}(l_0\mu/v_0\rho)^{1/2}$$

$$\pi_{d,2} = \mu\rho c_p/\rho k = \nu/\alpha = Pr$$

the solution being

$$hk^{-1}(l_0\mu/v_0\rho)^{1/2} = f(Pr)$$

which is generally written in teaching and research literature as a function of Nusselt and Reynolds numbers,

$$Nu_{l_0} Re_{l_0}^{-1/2} = f(Pr)$$

Nu_{l_0} and Re_{l_0} are not relevant dimensionless parameters from the perspective of DDA and, as a consequence, do not play an independent role in the solution (which is true). The dimensions of these numbers for the problem under study are:

$$[Nu_{l_0}] = [hl_0/k] = L_\parallel L_\perp^{-1}$$

$$[Re_{l_0}] = [v_0 l_0/\nu] = L_\parallel^2 L_\perp^{-2}$$

and the grouping "$Nu_{l_0} Re_{l_0}^{-1/2} \propto hk^{-1}(l_0\mu/v_0\rho)^{1/2}$" does indeed play an independent role in the solution.

Finally, if the unsuitable dimensional base ($L_\parallel, L_\perp, S_\perp, Q, T, \theta$) is used, a new dimensionless group will appear, namely the Eckert number,

$$Ec = \rho v_0^2/(\rho c_p)\Delta\theta$$

whose physical meaning is associated to the conversion of mechanical energy to thermal energy (or vice versa) which does not take place in this problem. So, the solution given by this base would be incorrect.

The asymptotic case of $\delta_t \gg \delta_v$ or $Pr \ll 1$. This hypothesis applies to high conductivity liquids, such as metals. The velocity boundary layer is very small compared with the thermal boundary layer, so that in most of the region where heat transfer occurs the fluid velocity is the non-disturbed velocity, v_0. In consequence, friction and inertial forces in this region are negligible. From this and eliminating μ and ρ in Table 3, the only dimensionless group that may be formed is

$$\pi_{d,1} = h(l_0/kv_0\rho c_p)^{1/2} = hk^{-1}(l_0\alpha/v_0)^{1/2}$$

which leads to the solution for the heat transfer coefficient

$$h \propto k(l_0\alpha/v_0)^{-1/2}$$

Text books [3,22] include integral and similarity solutions to this problem. The final result, given as a function of classical dimensionless Nusselt, Reynolds and Prandtl numbers, is

$$Nu_{l_0} \propto Re_{l_0}^{1/2} Pr^{1/2}$$

Table 2
Quantities for solving the shear stress and their dimensional exponents

	Relevant list				
	v_0	l_0	ρ	μ	τ
$L_\parallel$	1	1			1
$L_\perp$			−1	1	
$S_\perp$			−1	−1	−1
T	−1			−1	−2
M			1	1	1

Table 3
Quantities for solving the heat transfer coefficient and their dimensional exponents

	Relevant list							
	v_0	l_0	$\Delta\theta$	ρ	μ	k	ρc_p	h
$L_\parallel$	1	1						
$L_\perp$				−1	1	1	−1	
$S_\perp$				−1	−1	−1	−1	−1
Q						1	1	1
T	−1				−1	−1		−1
θ			1			−1	−1	−1
M				1	1			

As mentioned above, neither Nu_{l_0} nor Re_{l_0} is a relevant dimensionless parameter using DDA. On the contrary, it is obvious that Pr is a dimensionless number and that the product $Re_{l_0}Pr$, namely the Peclet number, is not a relevant dimensionless parameter. The expression $h \propto k(l_0\alpha/v_0)^{-1/2}$ may be converted to $Nu_{l_0} \propto Re_{l_0}^{1/2}Pr^{1/2}$ by introducing the kinematic viscosity. In this case, none of the numbers, Nu_{l_0}, Re_{l_0} or Pr, plays an independent role in the solution. Pr is associated to Re in the form of the product $Re_{l_0}Pr = Pe_{l_0}$ and this product, in turn, is associated to Nu_{l_0} in the form $Nu_{l_0}(Pr\,Re_{l_0})^{-1/2} = Nu_{l_0}Pe_{l_0}^{-1/2}$, which coincides with the only dimensionless group provided by the DDA.

As we have seen above in 4.1, among the mechanical aspects of the solid–liquid interface, the (hidden quantity) characteristic length l^* was the boundary velocity layer thickness. Also, as regards the thermal aspects, the dimensionless group deduced is $Nu_{l_0}Pe_{l_0}^{-1/2}$. This group may be written in the form $(h/k)(l_0Pe_{l_0}^{-1/2})$, where the characteristic length, δ_t, which makes the ratio h/k dimensionless is directly associated to the thermal boundary layer thickness, $\delta_t \propto l_0Pe_{l_0}^{-1/2}$. In fact, the group $(h/k)(l_0Pe_{l_0}^{-1/2})$ may be considered as a modified Nusselt number, $h\delta_t/k$. This conclusion derived from DDA is not presented in the research literature or in text books and handbooks. In short, DDA provides the correct order of magnitude for the heat transfer coefficient and clearly describes the role played (or not) by the well known classical numbers.

The asymptotic case $\delta_v \gg \delta_t$ or $Pr \gg 1$. This hypothesis applies to heavy, high viscosity oils. The velocity boundary layer is much thicker than the thermal boundary layer (very close to the plate). In consequence, the fluid velocities and accelerations are very small in this layer and friction and inertial effects are negligible, meaning that ρ and μ can be eliminated from the relevant list (Table 4).

In contrast with the former case, the non-disturbed velocity v_0 must not be included in the relevant list since v_0 does not characterize the movement in the small region of the thermal boundary layer. The characteristic velocity within this region must be a small fraction of v_0, namely $v^* = Cv_0$, with $C \ll 1$, a dimensionless factor. In short, the relevant list of quantities that defines the order of magnitude of δ_t is shown in Table 4 together with the dimensional exponents.

The solution for δ_t is

$$\delta_t \propto \left(l_0k/\rho c_p v^*\right)^{1/2} = \left(l_0\alpha/v^*\right)^{1/2}$$

The set of quantities to determine the heat transfer coefficient, h, is shown in Table 5. The only dimensionless group that may be formed is

$$\pi_{d,1} = h\left(l_0/kv^*\rho c_p\right)^{1/2} = hk^{-1}\left(l_0\alpha/v^*\right)^{1/2}$$

a kind of Nusselt number associated to δ_t or v^*. The order of magnitude of h is

$$h \propto k\left(l_0\alpha/v^*\right)^{-1/2}$$

The above results represent the contribution of DDA to determining the quantities δ_t and h. The new hypothesis assumed by many texts is that the numerical value of C is close to δ_t/δ_v. Based on this hypothesis, the substitution of $v^* = (\delta_t/\delta_v)v_0$ in the expressions of δ_t and h, gives

$$\delta_t \propto (l_0\alpha\delta_v/v_0)^{1/3}$$

$$h \propto k(l_0\alpha\delta_v/v_0)^{-1/3}$$

Now, substituting δ_v from 4.1 in these expressions, δ_t and h may be definitively written as functions of the quantities of the problem in the form

$$\delta_t \propto l_0\left(\mu k^2\rho^{-1}v_0^{-3}(\rho c_p)^{-2}l_0^{-3}\right)^{1/6}$$

$$h \propto (k/l_0)\left(\mu k^2\rho^{-1}v_0^{-3}(\rho c_p)^{-2}l_0^{-3}\right)^{-1/6}$$

These relations may be written (as is usual in the literature) in terms of the classical known numbers (Nu_{l_0}, Re_{l_0} and Pr) in the forms:

$$\delta_t \propto l_0Re_{l_0}^{-1/2}Pr^{-1/3}$$

$$Nu_{l_0} \propto Pr^{1/3}Re_{l_0}^{1/2}$$

Nevertheless, these final formulae say nothing relevant since, on the one hand, Nu_{l_0}, Pr and Re_{l_0} are not really dimensionless parameters (from the perspective of DDA) and, on the other, they do not play an independent (separate) role in this asymptotic case.

Finally, it is interesting to point out again that the expression $l_0Re_{l_0}^{-1/2}Pr^{-1/3}$ is the length that makes the ratio h/k dimensionless, that is, the suitable length to form a discriminated dimensionless Nusselt number.

Table 4
Quantities for solving δ_t and their dimensional exponents. $\delta_v \gg \delta_t$ or $Pr \gg 1$

	Relevant list					
	v^*	l_0	$\Delta\theta$	k	ρc_p	δ_t
$L_\parallel$	1	1				
$L_\perp$				1	−1	1
$S_\perp$				−1	−1	
Q				1	1	
T	−1			−1		
θ			1	−1	−1	
M						

Table 5
Quantities for solving h and their dimensional exponents. $\delta_v \gg \delta_t$ or $Pr \gg 1$

	Relevant list					
	v^*	l_0	$\Delta\theta$	k	ρc_p	h
$L_\parallel$	1	1				
$L_\perp$				1	−1	
$S_\perp$				−1	−1	−1
Q				1	1	1
T	−1			−1		−1
θ			1	−1	−1	−1
M						

5. Discriminated dimensional analysis and scale analysis

In our opinion, the relationship between SA and DDA is more than apparent although the formalism by which each technique is applied is quite different. While SA starts from the differential equations of the problem derived from the assumed hypothesis, DDA leaves the mathematical model aside. Nevertheless, an insight into the problem and its hypothesis is essential for precisely describing the relevant list of quantities that take part in the problem. Taking into account (or not) an addend in the differential equation of the problem in order to establish a balance in SA is equivalent to introducing or deleting from the relevant list the characteristic parameter associated to that addend in DDA. For example, if the viscous forces are negligible in the balance of forces, the viscosity is deleted from the relevant list. SA takes the simplified Navier–Stokes (momentum) equation for a flat plate under forced convection (which assumes the existence of a thin boundary layer proposed by Prandtl, δ_v) and, by manipulating this equation (referring the variables to characteristic values such as the length of the plate, l_0, the boundary layer thickness, δ_v, the non-disturbed velocity, v_0, etc.) the order of magnitude of δ_v is determined. This is the fundamental contribution of SA.

Formally, in the flat plate problem studied, SA takes two scale lengths and operates with this hypothesis in the simplified equation to look for the results. But, since the equations of the model are homogeneous, might we expect different results from those obtained with DDA?

The correct application of the DDA method requires a deep knowledge of the theory under study to establish both the appropriate dimensional basis of that theory and the precise variables of the relevant list. From this, the order of magnitude of the boundary layer thickness (a hidden quantity) can be obtained as a function of those quantities. In the application of DDA if the variables of the relevant list are well established, and if hidden quantities can be derived from them, such hidden quantities will have a precise physical meaning in the problem. Another simple example follows. For a finite 1-D isothermal flat plate of constant diffusivity, α, and thickness l_0, subjected to a step temperature at the boundary surfaces, $\Delta\theta$, the relevant list of the variables is set up by $\Delta\theta$, α and l_0. From these variables, a characteristic time (a hidden quantity) may be obtained, $\tau^* = l_0^2/\alpha$. This is the time required to finish the cooling or heating transient problem.

Finally, another important question concerns the connection between some of the classical numbers (*Re*, *Nu*, *Pe*, etc., which are not dimensionless in DDA) and the variable groups provided by SA and DDA. Firstly, neither SA nor DDA raises such a question. These numbers may be introduced into the solutions provided by SA and DDA to compare classical results of text books and the research literature. Both methods are able to assign the same real meanings to these classical numbers (different from the generally accepted meanings) and, from a fundamental point of view, neither method permits such these numbers to play an independent role in this kind of problem or gives them immediate physical meaning.

6. Conclusions

Discriminated dimensional analysis has been applied to demonstrate the dimensional homogeneity of the energy equation for incompressible fluids. In contrast with classical dimensional analysis, in the discriminated dimensional analysis, both spatial coordinates and mechanical or thermal properties (such as viscosity and thermal conductivity) have different dimensional equations, depending on the spatial directions. This fact provides a more precise insight into the meaning of the terms that take part in the energy equation.

The application of discriminated dimensional analysis to laminar forced convection on an isothermal flat plate, with negligible dissipation, leads to the final solution of the (i) velocity and thermal boundary layer thickness, (ii) drag coefficient and (iii) heat transfer coefficient. These solutions, which cannot be obtained by classical dimensional analysis, agree with those deduced by the application of numerical, analytical and scale analysis procedures.

In discriminated dimensional analysis, the classical numbers *Nu* and *Re* are not relevant dimensionless parameters and, consequently, they do not play an independent role in the solutions. Hence the combination of *Re* and *Pr* that appears in many solutions is artificial. Furthermore, discriminated dimensional analysis provides the characteristic lengths that take part in the “modified” Nusselt number, $Nu = hl^*/k$, to make it a “discriminated” relevant dimensionless parameter.

Finally, when the connection between scale analysis and discriminated dimensional analysis was studied in detail, it was seen that both methods lead to similar results despite the differences in the way they were applied.

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